CP violation in the effective action of the Standard Model

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ABSTRACT: Following a suggestion by Smit, the CP odd terms of the effective action of the Standard Model, obtained by integration of quarks and leptons, are computed to sixth order within a strict covariant derivative expansion approach. No other approximations are made. The final result so derived includes all Standard Model gauge fields and Higgs. Remarkably, at the order considered in this work, all parity violating contributions turn out to be zero. Non vanishing CP violating terms are obtained in the C-odd P-even sector. These are several orders of magnitude larger than perturbative estimates. Various unexpected regularities in the final result are noted.

KEYWORDS: Nonperturbative Effects, CP violation, Standard Model

Contents

1 Introduction

Whereas CPT symmetry is preserved of necessity in any theory with Lorentz invariance, locality and unitarity [1] no such mechanism is at work to preserve CP [2]. Nevertheless, CP is very weakly broken in the Standard Model [3, 4]. No breaking is detected in the strong sector, where the coupling constant of the operator $G_{\mu\nu}\tilde{G}^{\mu\nu}$ is compatible with zero [5]. In the electroweak sector no breaking is possible for less than three generations and even in this case the breaking would not occur in the presence of mass degeneration, as is the case in the

lepton sector for massless neutrinos [6]. So there is CP violation in the Standard Model but it is rather small, as compared to the maximal breaking of C or P in the electroweak sector.

The amount of CP breaking is relevant in early universe baryogenesis [7]. Baryon asymmetry generation is assumed to take place near the electroweak transition temperature, T of the order of $100\,\mathrm{GeV}$. At such high temperatures quark masses, except that of the top quark, can be considered small. This suggests to treat them as a perturbation. Since these masses follow from the Yukawa coupling of quarks to the Higgs field, this is equivalent to treat those Yukawa vertices perturbatively. However, as noted above, CP violation is elusive as no CP breaking term can be produced at low orders. The simplest such term is the Jarlskog determinant which appears at order twelve [8]

$$\Delta = J(m_u^2 - m_c^2)(m_c^2 - m_t^2)(m_t^2 - m_u^2)(m_d^2 - m_s^2)(m_s^2 - m_b^2)(m_b^2 - m_d^2), \qquad (1.1)$$

where J is the dimensionless Jarlskog invariant, constructed with the Cabibbo-Kobayashi Maskawa matrix. This term has dimension twelve and by dimensional counting it is usually assumed to enter in the effective action scaled by T^{12}. At the electroweak transition temperature the ratio Δ/T^{12} is extremely small, of the order of 10^{-19}. If this estimate is correct this poses a problem to account for baryogenesis using the Standard Model [9].

A simpleminded transcription of the above estimate to the zero temperature case can be achieved by simply replacing the scale T by the Higgs field condensate $v = 246\,\mathrm{GeV}$, $\Delta/v^{12} = 10^{-24}$. This is several orders of magnitude smaller than CP violation as measured in meson decays, where dimensionless parameters are of the order of 10^{-3} [10]. However, the direct use of the Jarlskog determinant is not justified at zero temperature, where the quark masses can no longer be treated perturbatively.

Smit [11] made the observation that a nonperturbative treatment would yield in a natural way much larger couplings for CP breaking operators, as such couplings would come out as rational functions (with logarithms) of the quark masses times the Jarlskog invariant. Specifically it was proposed to study the effective action of the Standard Model obtained after integration of the fermions in the theory. The full functional is, of course, beyond an exact computation and some type of classification and selection of the resulting terms is required. The proposal was then to organize the terms within a covariant derivative expansion, which being non perturbative has the potential of yielding a more reliable estimate for the couplings.

The study of the leading order terms in the abnormal parity sector was undertaken in [11]. This is the sector driven by the Levi-Civita pseudo-tensor and includes the Wess-Zumino-Witten term. In the Standard Model this is equivalent to the parity odd sector. The corresponding operators have dimension 4, counting only the dimension carried by the dynamical fields except the Higgs. Unfortunately no non vanishing contribution was found to fourth order, although CP breaking contributions were expected at dimension 6 [11]. Such dimension 6 operators, with non vanishing coupling, have been found in [12] where also the abnormal parity sector was studied. As expected, the CP violating term found is indeed sizable as compared to perturbative estimates.

The two calculations of CP violating terms in the Standard Model effective action just described are based on the technique introduced in [13, 14]. In this approach the full

effective action for a generic theory of fermions coupled to chiral gauge fields is computed within a strict derivative expansion, to a given order. [11] uses directly the result in [14] which holds to fourth order, and particularizes it to the Standard Model while selecting just the CP breaking terms. [12] carries out the same reduction from general to particular after extending the generic calculation to sixth order. This is done using the worldline formalism to deal with Dirac traces and momentum integration, as explained in [15], as an alternative to do the same thing with the techniques applied in [14].

In the present work we also undertake the calculation of the CP odd component of the Standard Model effective action at zero temperature derived by integration of quarks and leptons, and organize the terms so obtained by means of a covariant derivative expansion. However, unlike previous calculations, ours is carried out from scratch by applying the recently derived technique described in [16, 17]. The difference with previous approaches is that we particularize very early our treatment to the Standard Model and this allows us to select from the beginning terms which are candidate to break CP invariance and neglect irrelevant CP even terms. This is useful as CP breaking imposes very restrictive conditions and selects very few candidates. Another difference is that we consider terms of normal parity (i.e., P even and consequently C odd) as well as of abnormal parity (P odd, C even). Seemingly, in the literature, the CP breaking terms have been assumed to be of abnormal parity only. Perhaps this is because in the so called strong CP problem, the CP breaking terms involve the Levi-Civita pseudo-tensor in the gluon sector or γ_5 in the quark sector [18, 19]. Moreover, the simplest CP odd terms one can write for the effective action are also odd under intrinsic parity, e.g., $\text{tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu})$ [20], where $F_{\mu\nu}$ is constructed with the covariant derivative of the electroweak group $SU(2) \times U(1)$. In addition, the CP odd and P odd sector is of interest in the study of electric dipole moment. Nevertheless, it is perfectly possible to write down operators, constructed with the Standard Model gauge fields and the Higgs, having normal intrinsic parity and odd under CP transformations.

We carry out a detailed calculation including all Standard Model fields and all CP violating contributions to order six in the covariant derivative expansion. Diagrammatically, this consists of all one-loop Feynman graphs with fermions running on the loop and up to six gauge fields or derivatives (four-momenta) attached as external legs and any number of Higgs fields coupled to the quark or leptons. The result is given in the unitary gauge. The calculation presented is largely self-contained and some of the main conclusions obtained can be reached by a by-hand computation. This is the case for our most unexpected finding, namely, that there are CP violating terms of order six in the normal parity sector of the Standard Model, but all terms vanish in the abnormal parity sector. This is at variance with the result in [12]. It is not clear to us from where the discrepancy arises as the two calculations are conceptually similar although technically different. In any case we have double-checked our results to confirm this conclusion.

The paper is organized as follows. Section 2 summarizes a number of definitions, formulas and techniques relative to generic chiral gauge theories which will then be applied to the Standard Model. In section 3 we cast the fermionic sector of the Standard Model in the format previously described for generic chiral gauge theories. In section 4 features of CP violation at the level of the effective action are discussed. In section 5 the relevant mo-

mentum integrals which appear in the calculation, as well as the associated selection rules, are obtained. Section 6 discusses the covariant derivative expansion within our approach to the effective action of the Standard Model. In section 7 the chiral invariant approach devised for generic theories is applied to the Standard Model in a way that allows to easily remove irrelevant CP even terms. Section 8 reproduces the previous result in [11] verifying that there are no CP breaking terms driven by operators of order four in the abnormal parity sector, and this result is extended to the normal parity sector as well. In section 9 we present a preliminary calculation for the particular case where Higgs field derivatives are neglected. This is of interest as this covers the result obtained in [12]. The cancellation of the abnormal parity contribution is made manifest there within a transparent calculation. Section 10 presents the full result of our computation, eq. (10.2). Various surprising regularities in the result are noted. The form of the loop function controlling the CP breaking operator is discussed in section 11 and how it is affected by infrared enhancement in the physically relevant chiral limit. Our conclusions are summarized in section 12.

2 Chiral gauge fermions

In this section we collect important practical results relative to generic chiral gauge fermion theories which will be applied subsequently to the Standard Model.

2.1 The effective action

The Lagrangian describing the coupling of spin $1/2$ fermions (ψ) to chiral gauge fields $(V_{L,R})$ and spin zero fields (m_{LR}, m_{RL}) can be cast in the general form

$$\mathcal{L}(x) = \bar{\psi}(x)\, i\mathbf{D}(x)\, \psi(x)\,, \tag{2.1}$$

where[1]

$$i\mathbf{D}(x) = (i\slashed{\partial} - \slashed{V}_R(x))P_R + (i\slashed{\partial} - \slashed{V}_L(x))P_L - m_{LR}(x)P_R - m_{RL}(x)P_L, \tag{2.2}$$

and

$$P_R = \frac{1}{2}(1 + \gamma_5), \quad P_L = \frac{1}{2}(1 - \gamma_5), \tag{2.3}$$

and the external fields m_{LR}, m_{RL}, $V_{R,L}$ are matrices in some internal space. Unitarity requires

$$m_{LR}^\dagger(x) = m_{RL}(x), \quad V_{R,\mu}^\dagger(x) = V_{R,\mu}(x), \quad V_{L,\mu}^\dagger(x) = V_{L,\mu}(x). \tag{2.4}$$

It will prove convenient to write the Lagrangian in matricial form, namely,

$$\mathcal{L}(x) = (\bar{\psi}_L, \bar{\psi}_R) \begin{pmatrix} -m_{LR} & i\slashed{\partial} - \slashed{V}_L(x) \\ i\slashed{\partial} - \slashed{V}_R(x) & -m_{RL} \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \tag{2.5}$$

This form is useful to expose the action of chiral transformations.

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[1]We use Minkowskian signature $(+ - - -)$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\gamma_5 = +i\gamma^0\gamma^1\gamma^2\gamma^3$, $\epsilon_{0123} = +1$.

The integration of the fermions provides the effective action

$$Z = e^{i\Gamma} = \int D\bar{\psi}D\psi \, e^{i \int d^4x \, \bar{\psi}(x) \, i\mathbf{D} \, \psi(x)}, \tag{2.6}$$

which (modulo UV ambiguities) is given by

$$i\Gamma[m_{LR}, m_{RL}, V_R, V_L] = \operatorname{Tr} \log i\mathbf{D}. \tag{2.7}$$

The effective action can be decomposed into a normal parity component, Γ^+ (without Levi-Civita pseudo-tensor) and an abnormal parity component Γ^- (with the Levi-Civita pseudo-tensor):

$$\Gamma = \Gamma^+ + \Gamma^-. \tag{2.8}$$

$\Gamma^\pm$ are also even/odd, respectively, under the pseudo-parity transformation, which can be defined as the exchange of the labels LR in the external fields.

Among other symmetries, the effective action is invariant under the transformation

$$\begin{aligned}
m_{LR}(x) &\to m_{LR}^*(\tilde{x}), & m_{RL}(x) &\to m_{RL}^*(\tilde{x}), \\
V_{R,\mu}(x) &\to -\pi_\mu{}^\nu V_{R,\nu}^*(\tilde{x}), & V_{L,\mu}(x) &\to -\pi_\mu{}^\nu V_{L,\nu}^*(\tilde{x}),
\end{aligned} \tag{2.9}$$

where $\pi^\mu{}_\nu = \operatorname{diag}(1, -1, -1, -1)$ and $\tilde{x} = (x^0, -\vec{x}) = \pi x$. This represents a CP transformation which we shall call *full CP transformation* to distinguish it from the physical one (which only acts on dynamical fields and not on parameters of the Lagrangian).[2] The important property of this transformation is that it does not mix different chiral sectors.

2.2 Euclidean space

For convenience we shall work in Euclidean space, reverting to Minkowskian space at the end.

The Euclidean metric is $\delta_{\mu\nu}$, so we put Euclidean indices as subindices. The pass from Minkowskian to Euclidean variables is achieved by the replacements[3]

$$\begin{aligned}
(x^0, x^i) &\to (-ix^0, x^i), \\
\psi(x) &\to \psi(x), & \bar{\psi}(x) &\to \bar{\psi}(x), \\
m_{LR}(x) &\to m_{LR}(x), & m_{RL}(x) &\to m_{RL}(x), \\
V_{R,0}(x) &\to V_{R,0}(x), & V_{L,0}(x) &\to V_{L,0}(x), \\
V_{R,i}(x) &\to -iV_{R,i}(x), & V_{L,i}(x) &\to -iV_{L,i}(x), \\
\gamma^0 &\to \gamma_0, & \gamma^i &\to i\gamma_i, & \gamma_5 \to \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3.
\end{aligned} \tag{2.10}$$

These replacements imply $\mathcal{L}(x) \to -\mathcal{L}(x)$ and so

$$e^{i\Gamma} = \int D\bar{\psi}D\psi \, e^{i \int d^4x \, \mathcal{L}} \to e^{-\Gamma} = e^{-\int d^4x \mathcal{L}(x)} \tag{2.11}$$

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[2]Likewise, *full parity* is also a symmetry of the effective action. It consists of exchanging LR labels and simultaneously $x \to \tilde{x}$. Pseudo-parity, which can be defined as any of these two transformations without the other, is not a symmetry for general background field configurations.

[3]This corresponds to $\eta_4 = 1$ in ref. [14].

with

$$\mathcal{L}(x) = \bar{\psi}(x)\mathbf{D}\,\psi(x)\,,\tag{2.12}$$

and

$$\mathbf{D} = (\slashed{\partial} + \slashed{V}_R(x))P_R + (\slashed{\partial} + \slashed{V}_L(x))P_L + m_{LR}(x)P_R + m_{RL}(x)P_L.\tag{2.13}$$

Also, in Euclidean space

$$\Gamma[m_{LR}, m_{RL}, V_R, V_L] = \mathrm{Tr}\log\mathbf{D}\,.\tag{2.14}$$

With the above prescriptions, in Euclidean space unitarity becomes

$$m_{LR}^\dagger(x) = m_{RL}(x), \quad V_{R,\mu}^\dagger(x) = -V_{R,\mu}(x), \quad V_{L,\mu}^\dagger(x) = -V_{L,\mu}(x),\tag{2.15}$$

while the full CP transformation becomes

$$m_{LR}(x) \to m_{LR}^*(\tilde{x}), \quad m_{RL}(x) \to m_{RL}^*(\tilde{x}), \quad V_{R,\mu}(x) \to \pi_{\mu\nu}V_{R,\nu}^*(\tilde{x}), \quad V_{L,\mu}(x) \to \pi_{\mu\nu}V_{L,\nu}^*(\tilde{x}),\tag{2.16}$$

with $\pi_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $\tilde{x} = (x^0, -\vec{x}) = \pi x$.

An important property of the effective action in Euclidean space is that the normal parity component, Γ^+, is real, and the abnormal parity component, Γ^-, is purely imaginary [21]. This property is a consequence of unitarity and holds at the non perturbative level.

2.3 Chiral invariant approach to the effective action

The (Euclidean) Lagrangian (2.12) is invariant under local chiral transformations:

$$\mathbf{D} \longrightarrow \mathbf{D}^\Omega = \begin{pmatrix} \Omega_L^{-1} & 0 \\ 0 & \Omega_R^{-1} \end{pmatrix} \begin{pmatrix} m_{LR} & \slashed{D}_L \\ \slashed{D}_R & m_{RL} \end{pmatrix} \begin{pmatrix} \Omega_R & 0 \\ 0 & \Omega_L \end{pmatrix}.\tag{2.17}$$

However, as is well known, the corresponding effective action $\Gamma[m_{LR}, m_{RL}, V_R, V_L]$ displays an anomalous variation under chiral transformations [22–24]. The anomaly has a universal, geometrical, form and is saturated by the gauged Wess-Zumino-Witten (WZW) term, $\Gamma_{\mathrm{gWZW}}[m_{LR}, m_{RL}, V_R, V_L]$, which also has a known geometrical form [25, 26]. This means that if the WZW term is subtracted from the effective action, the remainder, Γ_c, is chiral invariant:

$$\Gamma = \Gamma_{\mathrm{gWZW}} + \Gamma_c.\tag{2.18}$$

Let us note that Γ_{gWZW} contributes only to the abnormal parity component. Therefore, $\Gamma^+ = \Gamma_c^+$.

Recently, it has been shown that the remainder Γ_c can be expressed in a convenient form in which chiral invariance is manifest [16]. Indeed, let

$$K = m_{LR}m_{RL} - \slashed{D}_L m_{RL}^{-1}\slashed{D}_R m_{RL},\tag{2.19}$$

where $D_{L,\mu} = \partial_\mu + V_{L,\mu}$ and $D_{R,\mu} = \partial_\mu + V_{R,\mu}$, and so K is a second order differential operator. (Note that $\not{D}_L m_{RL}^{-1} \not{D}_R m_{RL}$ stands for the product of four consecutive operators.) Then[4]

$$
\begin{aligned}
\Gamma^+ &= -\frac{1}{2}\operatorname{Re}\operatorname{Tr}(\log K), \\
\Gamma^- &= -\frac{1}{2}i\operatorname{Im}\operatorname{Tr}(\gamma_5 \log K) + \Gamma_{\text{gWZW}}.
\end{aligned}
\tag{2.20}
$$

2.4 The derivative expansion

The functional trace in (2.14) can not be computed in closed form in general. This suggests to use instead some systematic expansion to address its determination. In the derivative expansion scheme the effective action contributions are classified according to the number of covariant derivatives they carry. In this counting the spin zero fields, m_{LR}, m_{RL}, count as order zero whereas the derivatives ∂_μ or gauge fields, $V_{L,\mu}$, $V_{R,\mu}$, count as first order. Technically, this means to consider the family of Dirac operators

$$
\mathbf{D}_t = (\not{\partial} + t\not{V}_R(tx))P_R + (\not{\partial} + t\not{V}_L(tx))P_L + m_{LR}(tx)P_R + m_{RL}(tx)P_L,
\tag{2.21}
$$

and then expand the corresponding effective action in powers of the parameter t. After extraction of a global factor $1/t^d$ in d space-time dimensions, the terms of order n contain n covariant derivatives (or gauge fields). At the diagrammatic level, a term of order n represents a one-loop Feynman graph with any number of scalar fields and n gauge fields as external legs, all of them at zero four-momentum, or less gauge fields as external legs and correspondingly more powers of the external momenta. We emphasize that gauge fields are assimilated to derivatives in such a way that they are of the same order. This ensures preservation of gauge invariance and as a consequence each order of the derivative expansion of the effective action is separately invariant under gauge transformations.

Several remarks can be made:

  i) In even-dimensional space-times and at zero temperature, the derivative expansion of the effective action contains even orders only.

 ii) The abnormal parity sector starts at order d in d dimensions, since it contains the Levi-Civita pseudo-tensor.

iii) In d dimensions, the terms beyond order d are ultraviolet (UV) convergent, and so free from UV ambiguities introduced by the renormalization.

 iv) Γ_{gWZW} contains only terms of order d (in d dimensions). I.e., within the derivative expansion, the WZW term vanishes at all orders beyond the lowest order one. This means that $\Gamma^- = \Gamma_c^-$ except at lowest order. And

  v) within the derivative expansion, the Minkowskian version of the effective action is real in the normal parity and in the abnormal parity sectors. The derivative expansion

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[4]In the notation of [16], K here is K_L there, and we have used that K_R and $K_L^\dagger$ are related by a similarity transformation.

is an expansion around small external four-momentum, and so it can not reach the analytical cuts related to particle production.

2.5 The method of symbols

A convenient technique to carry out calculations within the derivative expansion is the method of symbols [27, 28]. The method has been extended to curved space-time [29, 30] and finite temperature [31, 32].

This method will be used below in our calculation of the effective action.

For any pseudo-differential operator $\hat{f}$ of the form $f(D, M)$, constructed with covariant derivatives D_μ and external fields $M(x)$ (all this non abelian in general) the method of symbols states, for the diagonal matrix elements of $\hat{f}$,

$$\langle x|\hat{f}|x\rangle = \int \frac{d^d p}{(2\pi)^d} \langle x|f(D + p, M)|0\rangle \tag{2.22}$$

(d is the dimension of the space of x). $|0\rangle$ represents the constant state $\langle x|0\rangle = 1$, so that $\partial_\mu|0\rangle = 0$. For notational convenience, here and in what follows, we use a purely imaginary momentum $p_\mu = ik_\mu$ (k_μ real), but p^2 denotes $-p_\mu p_\mu = k^2$ and $d^d p = d^d k$. The matrix element $\langle x|f(D + p, M)|0\rangle$ just coincides with the standard symbol of $\hat{f}$, as defined in the theory of pseudo-differential operators [33].

Invariance under the shift $p_\mu \to p_\mu + a_\mu$ in the momentum integral (2.22) implies that D_μ can appear only in the form $[D_\mu, \ ]$ in the final integrated expression. This ensures gauge covariance of the right-hand side of (2.22), consistently with the obvious gauge covariance of the left-hand side.

From (2.22) the derivative expansion is easily obtained just by formally expanding in powers of D_μ.[5] The final step is to move all derivatives to the right (derivating everything in passing as dictated by Leibniz's rule) and to verify that terms with derivatives at the right (which would break gauge invariance) vanish after carrying out the momentum integration.

The method of symbols then provides the (functional) trace of $\hat{f}$ as

$$\text{Tr}(\hat{f}) = \int \frac{d^d x \, d^d p}{(2\pi)^d} \text{tr} \, f(D + p, M). \tag{2.23}$$

(The brackets $\langle x| \ |0\rangle$ are usually omitted.)

The method of symbols just described is simple enough but has the drawback that gauge invariance is not manifest during the calculation. An alternative technique, which we shall also employ in this work, is that of *covariant* symbols [30, 34], which has the virtue of being manifestly covariant from the beginning. Quite simply, the method consists in applying a similarity transformation in (2.22), which changes nothing:

$$\langle x|\hat{f}|x\rangle = \int \frac{d^d p}{(2\pi)^d} e^{-D_\mu \partial/\partial p_\mu} f(D + p, M) e^{D_\nu \partial/\partial p_\nu}. \tag{2.24}$$

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[5]p_μ is the momentum running in the quantum loop and is of order zero in the derivative expansion. On the other hand D_μ now acts only on the other external fields present in $\hat{f}$ and counts as first order. Direct expansion in powers of D_μ in $f(D, M)$ (i.e. before applying the method of symbols) would not produce UV convergent integrals and would not correspond to the derivative expansion, as D_μ would contain not only the momenta of the external fields but also that of the quantum field running in the loop.

This can be seen to be equivalent to

$$\langle x|\hat{f}|x\rangle = \int \frac{d^d p}{(2\pi)^d} f(\bar{D}, \bar{M}) \,, \tag{2.25}$$

where $\bar{D}_\mu$ and $\bar{M}$ are manifestly gauge covariant:

$$\bar{M} = M - [D_\alpha, M]\frac{\partial}{\partial p_\alpha} + \frac{1}{2!}[D_\alpha, [D_\beta, M]]\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p_\beta} - \cdots$$

$$\bar{D}_\mu = p_\mu - \frac{1}{2!}[D_\alpha, D_\mu]\frac{\partial}{\partial p_\alpha} + \frac{2}{3!}[D_\alpha, [D_\beta, D_\mu]]\frac{\partial}{\partial p_\alpha}\frac{\partial}{\partial p_\beta} - \cdots \tag{2.26}$$

and these series are truncated at the desired order in the derivative expansion.

3 Standard Model

3.1 Fermion sector of the Standard Model

We can apply the previous general results to the Standard Model (SM) for quarks and leptons coupled to gauge fields and Higgs [10, 35]. For quarks (and in Minkowski space)

$$\mathcal{L}_{\text{SM,q}}(x) = \bar{q}(x)\, i\mathbf{D}_{\text{SM,q}}\, q(x)\,. \tag{3.1}$$

In the notation of (2.5) the quark field, $q(x)$, is a column matrix in the space $\mathcal{H}_{LR} \otimes \mathcal{H}_{ud} \otimes \mathcal{H}_{\text{gen}} \otimes \mathcal{H}_{\text{color}} \otimes \mathcal{H}_{\text{Dirac}}$, where $\mathcal{H}_{LR} = \mathcal{H}_L \oplus \mathcal{H}_R$ (chirality) has dimension two, $\mathcal{H}_{ud} = \mathcal{H}_u \oplus \mathcal{H}_d$ (u or d quark type) has dimension two, $\mathcal{H}_{\text{gen}} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ (generation) has dimension three, $\mathcal{H}_{\text{color}}$ has dimension three and $\mathcal{H}_{\text{Dirac}}$ has dimension four.

$$q(x) = \begin{pmatrix} u_R \\ d_R \\ u_L \\ d_L \end{pmatrix}, \quad \bar{q}(x) = \left(\bar{u}_L, \bar{d}_L, \bar{u}_R, \bar{d}_R \right). \tag{3.2}$$

Here only the LR and ud spaces are explicit while generation, color and Dirac indices have been left implicit. Likewise, in the unitary gauge, the Dirac operator takes the form

$$i\mathbf{D}_{\text{SM,q}} =$$
$$\begin{pmatrix} -\frac{1}{\sqrt{2}}\phi Y_u & 0 & i\slashed{\partial} - \frac{1}{2}g\slashed{\tilde{W}}_3 - \frac{1}{6}g'\slashed{\tilde{B}} - \frac{1}{2}g_s\lambda_a\slashed{\tilde{G}}_a & -\frac{1}{\sqrt{2}}g\slashed{\tilde{W}}^+ \\ 0 & -\frac{1}{\sqrt{2}}\phi Y_d & -\frac{1}{\sqrt{2}}g\slashed{\tilde{W}}^- & i\slashed{\partial} + \frac{1}{2}g\slashed{\tilde{W}}_3 - \frac{1}{6}g'\slashed{\tilde{B}} - \frac{1}{2}g_s\lambda_a\slashed{\tilde{G}}_a \\ i\slashed{\partial} - \frac{2}{3}g'\slashed{\tilde{B}} - \frac{1}{2}g_s\lambda_a\slashed{\tilde{G}}_a & 0 & -\frac{1}{\sqrt{2}}\phi Y_u^\dagger & 0 \\ 0 & i\slashed{\partial} + \frac{1}{3}g'\slashed{\tilde{B}} - \frac{1}{2}g_s\lambda_a\slashed{\tilde{G}}_a & 0 & -\frac{1}{\sqrt{2}}\phi Y_d^\dagger \end{pmatrix}. \tag{3.3}$$

$Y_{u,d}$ are 3×3 matrices in generation space which denote the Yukawa couplings of the quarks with the Higgs field $\phi(x)$. $\tilde{G}_{a,\mu}$ are the gluon fields, with coupling constant g_s, and λ_a the Gell-Mann matrices in color space. $\tilde{B}_\mu$ is the U(1) weak hypercharge gauge field, with coupling constant g', $\tilde{W}_\mu^\pm$ and $\tilde{W}_{3,\mu}$ are the SU(2) weak isospin gauge fields, with

coupling constant g. All matrices are the identity in generation space, except $Y_{u,d}$, the identity in color space, except λ_a, and the identity in Dirac space, except γ^μ.

The gauge fields shown are the canonical ones. For convenience we shall absorb the couplings in the fields, and write the Dirac operator in the form

$$i\mathbf{D}_{\mathrm{SM,q}} = \begin{pmatrix} -\frac{\phi}{v}M_u & 0 & i\slashed{D}_u - \slashed{Z} - \slashed{G} & -\slashed{W}^+ \\ 0 & -\frac{\phi}{v}M_d & -\slashed{W}^- & i\slashed{D}_d + \slashed{Z} - \slashed{G} \\ i\slashed{D}_u - \slashed{G} & 0 & -\frac{\phi}{v}M_u^\dagger & 0 \\ 0 & i\slashed{D}_d - \slashed{G} & 0 & -\frac{\phi}{v}M_d^\dagger \end{pmatrix}. \tag{3.4}$$

In this expression v is the vacuum expectation value of the Higgs field after spontaneous symmetry breaking, and $M_{u,d}$ the complex quark mass matrices,

$$v = \langle \phi \rangle, \quad M_u = \frac{1}{\sqrt{2}}vY_u, \quad M_d = \frac{1}{\sqrt{2}}vY_d. \tag{3.5}$$

$$G_\mu = \frac{1}{2}g_s\lambda_a\tilde{G}_{a,\mu}. \tag{3.6}$$

Also

$$W_\mu^\pm = \frac{1}{\sqrt{2}}g\tilde{W}_\mu^\pm, \quad Z_\mu = \frac{1}{2}g\tilde{W}_{3,\mu} - \frac{1}{2}g'\tilde{B}_\mu = \frac{1}{2}\frac{g}{\cos\theta_W}\tilde{Z}_\mu, \tag{3.7}$$

where $\tilde{Z}_\mu(x)$ is the canonical field of the Z^0 boson and θ_W the weak angle. In addition, we have introduced the covariant derivatives

$$D_{u,\mu} = \partial_\mu + iA_{u,\mu}, \quad D_{d,\mu} = \partial_\mu + iA_{d,\mu}, \tag{3.8}$$

with

$$A_{u,\mu} = \frac{2}{3}g'\tilde{B}_\mu, \quad A_{d,\mu} = -\frac{1}{3}g'\tilde{B}_\mu. \tag{3.9}$$

For leptons

$$\mathcal{L}_{\mathrm{SM,l}}(x) = \bar{l}(x)\,i\mathbf{D}_{\mathrm{SM,l}}\,l(x). \tag{3.10}$$

The lepton field $l(x)$ belongs to the space $\mathcal{H}_{LR} \otimes \mathcal{H}_{\nu e} \otimes \mathcal{H}_{\mathrm{gen}} \otimes \mathcal{H}_{\mathrm{Dirac}}$. For convenience it includes a spurious right-handed neutrino to achieve greater similarity with the quark case. In matrix form the fields are organized as follows

$$l(x) = \begin{pmatrix} \nu_R \\ e_R \\ \nu_L \\ e_L \end{pmatrix}, \quad \bar{l}(x) = \left(\bar{\nu}_L, \bar{e}_L, \bar{\nu}_R, \bar{e}_R \right), \tag{3.11}$$

and the Dirac operator takes the form (we assume massless neutrinos throughout)

$$i\mathbf{D}_{\mathrm{SM,l}} = \begin{pmatrix} 0 & 0 & i\slashed{\partial} - \frac{1}{2}g\slashed{\tilde{W}}_3 + \frac{1}{2}g'\slashed{\tilde{B}} & -\frac{1}{\sqrt{2}}g\slashed{\tilde{W}}^+ \\ 0 & -\frac{1}{\sqrt{2}}\phi Y_e & -\frac{1}{\sqrt{2}}g\slashed{\tilde{W}}^- & i\slashed{\partial} + \frac{1}{2}g\slashed{\tilde{W}}_3 + \frac{1}{2}g'\slashed{\tilde{B}} \\ i\slashed{\partial} & 0 & 0 & 0 \\ 0 & i\slashed{\partial} + g'\slashed{\tilde{B}} & 0 & -\frac{1}{\sqrt{2}}\phi Y_e^\dagger \end{pmatrix}. \tag{3.12}$$

The right-handed neutrino is completely decoupled. As for the quarks, we find it convenient to rewrite the same matrix as

$$
i\mathbf{D}_{\mathrm{SM,l}} = \begin{pmatrix} 0 & 0 & i\slashed{\partial} - \slashed{Z} & -\slashed{W}^+ \\ 0 & -\frac{\phi}{v}M_e & -\slashed{W}^- & i\slashed{D}_e + \slashed{Z} \\ i\slashed{\partial} & 0 & 0 & 0 \\ 0 & i\slashed{D}_e & 0 & -\frac{\phi}{v}M_e^\dagger \end{pmatrix},
\tag{3.13}
$$

where we have introduced the new covariant derivative

$$
D_{e,\mu} = \partial_\mu + iA_{e,\mu}, \quad A_{e,\mu} = -g'\tilde{B}_\mu.
\tag{3.14}
$$

For subsequent use, we introduce the derivatives

$$
\begin{aligned}
W^+_{\mu\nu} &= D_{u,\mu}W^+_\nu - W^+_\nu D_{d,\mu} = [\partial_\mu, W^+_\nu] + i(A_{u,\mu} - A_{d,\mu})W^+_\nu, \\
W^-_{\mu\nu} &= D_{d,\mu}W^-_\nu - W^-_\nu D_{u,\mu} = [\partial_\mu, W^-_\nu] - i(A_{u,\mu} - A_{d,\mu})W^-_\nu, \\
F^{u,d}_{\mu\nu} &= -i[D^{u,d}_\mu, D^{u,d}_\nu] = [\partial_\mu, A^{u,d}_\nu] - [\partial_\nu, A^{u,d}_\mu].
\end{aligned}
\tag{3.15}
$$

Note that the fields $W^\pm_{\mu\nu}$ are *not* antisymmetric in μ,ν. Also note that the analogous construction in the lepton sector gives exactly the same result as for the quark sector, namely, $W^\pm_{\mu\nu} = [\partial_\mu, W^\pm_\nu] \pm ig'\tilde{B}_\mu W^\pm_\nu$. In fact, using the relation

$$
A^{\mathrm{e.m.}}_\mu = g'\tilde{B}_\mu + 2\sin^2\theta_W Z_\mu \quad \text{(photon field)},
\tag{3.16}
$$

one finds

$$
W^\pm_{\mu\nu} = D^{\mathrm{e.m.}}_\mu W^\pm_\nu \mp i2\sin^2\theta_W Z_\mu W^\pm_\nu,
\tag{3.17}
$$

which is covariant under $\mathrm{U}_{\mathrm{e.m.}}(1)$, the remaining gauge freedom in the unitary gauge, apart from $\mathrm{SU}_{\mathrm{color}}(3)$.

3.2 Euclidean space

We apply to the SM fields the prescriptions given in section 2.2 to go from Minkowskian to Euclidean space, and this yields

$$
\mathbf{D}_{\mathrm{SM,q}} = \begin{pmatrix} \frac{\phi}{v}M_u & 0 & \slashed{D}_u + \slashed{Z} + \slashed{G} & \slashed{W}^+ \\ 0 & \frac{\phi}{v}M_d & \slashed{W}^- & \slashed{D}_d - \slashed{Z} + \slashed{G} \\ \slashed{D}_u + \slashed{G} & 0 & \frac{\phi}{v}M_u^\dagger & 0 \\ 0 & \slashed{D}_d + \slashed{G} & 0 & \frac{\phi}{v}M_d^\dagger \end{pmatrix}
\tag{3.18}
$$

$$
\mathbf{D}_{\mathrm{SM,l}} = \begin{pmatrix} 0 & 0 & \slashed{\partial} + \slashed{Z} & \slashed{W}^+ \\ 0 & \frac{\phi}{v}M_e & \slashed{W}^- & \slashed{D}_e - \slashed{Z} \\ \slashed{\partial} & 0 & 0 & 0 \\ 0 & \slashed{D}_e & 0 & \frac{\phi}{v}M_e^\dagger \end{pmatrix}
\tag{3.19}
$$

with

$$
D_{u,\mu} = \partial_\mu + A_{u,\mu}, \quad D_{d,\mu} = \partial_\mu + A_{d,\mu}, \quad D_{e,\mu} = \partial_\mu + A_{e,\mu}.
\tag{3.20}
$$

Also

$$W_{\mu\nu}^+ = D_{u,\mu}W_\nu^+ - W_\nu^+ D_{d,\mu}, \quad W_{\mu\nu}^- = D_{d,\mu}W_\nu^- - W_\nu^- D_{u,\mu},$$
$$F_{\mu\nu}^{u,d} = [D_\mu^{u,d}, D_\nu^{u,d}]. \tag{3.21}$$

The c-number fields Z_μ, $A_{u,\mu}$, $A_{d,\mu}$, $A_{e,\mu}$ and $F_{\mu\nu}^{u,d}$ are purely imaginary, while $(W_\mu^+)^* = -W_\mu^-$ and $(W_{\mu\nu}^+)^* = -W_{\mu\nu}^-$. The matrix field G_μ is antihermitian. The field ϕ is a real c-number.

In what follows we shall work in Euclidean space, until section 10, where we return to Minkowskian space to display the results.

3.3 ud-parity

In the SM only two of the four fields Z_μ, $A_{u,\mu}$, $A_{d,\mu}$, and $A_{e,\mu}$ are independent (and $F_{\mu\nu}^d$ is proportional to $F_{\mu\nu}^u$), however, it proves useful to carry out the calculation for the "extended model" where Z_μ, $A_{u,\mu}$, $A_{d,\mu}$, and $A_{e,\mu}$ are independent fields, and also M_u and M_d are regarded as (x-independent) variables, as this procedure provides simpler expressions. The quark sector with generic A_u and A_d enjoys a $\mathrm{U}_u(1) \times \mathrm{U}_d(1)$ symmetry which in the SM reduces to $\mathrm{U}_{\mathrm{e.m.}}(1)$. Moreover, for the extended model a symmetry becomes apparent under the exchange of labels u and d in the quark sector, namely,

$$M_u \leftrightarrow M_d, \quad W^\pm \leftrightarrow W^\mp, \quad A_u \leftrightarrow A_d, \quad Z \leftrightarrow -Z, \quad G \leftrightarrow G, \quad \phi \leftrightarrow \phi. \tag{3.22}$$

This corresponds to a similarity transformation of $\mathbf{D}_{\mathrm{SM,q}}$ as given in (3.18) and so it leaves the (quark sector) effective action unchanged when expressed in terms of generic M_u, M_d, A_u, A_d and Z (as well a $W^\pm$ and ϕ). This symmetry, which we shall call ud-parity, is not supported by the SM but it will be present in our calculation and this will become useful later.

4 CP violation

As noted, the full CP transformation in (2.16) is a symmetry of the effective action functional. It is instructive to see this in detail. First, note that, in Euclidean space and in four dimensions, the definition of γ_5 does not contain an imaginary unit i, and so no complex numbers are generated in the functional Γ after taking Dirac traces (nor there are any other i's in the Dirac operator or the definition of Γ, cf. (2.14)). As a consequence, when the background fields are replaced by their complex conjugated, Γ also becomes complex conjugated. That is, the (real) normal parity component, Γ^+ is unchanged whereas the (purely imaginary) abnormal parity component, Γ^- changes to $-\Gamma^-$. On the other hand, the transformation involving $(x^0, \vec{x}) \to (x^0, -\vec{x})$ leaves invariant Γ^+ since it does not contain $\epsilon_{\mu\nu\alpha\beta}$, but changes the sign of Γ^-. In this way the complete effective action is left invariant under full CP transformations.

Consider now the (Euclidean) *physical CP transformation* in the SM. The transformation acts on the dynamical fields, namely, G_a, ϕ, $W^\pm$, Z, and $A_{u,d,e}$, but the constants $M_{u,d,e}$ are unchanged and in general this will no leave the effective action invariant. This

allows to classify the contributions to the effective action in two types according to whether they are even or odd under physical CP:

$$\Gamma = \Gamma_+ + \Gamma_- \,. \tag{4.1}$$

(This classification is not to be confused with the separation (2.8) into terms even and odd under pseudo-parity.)

In view of the fact that the full CP transformation leaves Γ invariant, it follows that the physical CP transformation has the same effect on the effective action as the transformation (v is real):

$$M_{u,d,e} \to M^*_{u,d,e} \,. \tag{4.2}$$

This implies the well known result that if the Yukawa couplings were real, or equivalent to real, there would be no CP violation. This is automatically the case in the lepton sector but not in the quark sector for three or more generations [6]. Indeed, for arbitrary complex matrices M_u and M_d one can write

$$M_u = \mathcal{A}_{u,L}^{-1} m_u \mathcal{A}_{u,R} \,, \quad M_d = \mathcal{A}_{d,L}^{-1} m_d \mathcal{A}_{d,R} \,, \tag{4.3}$$

where the matrices m_u, m_d are diagonal and non negative and $\mathcal{A}_{u,L}$, $\mathcal{A}_{u,R}$, $\mathcal{A}_{d,L}$, $\mathcal{A}_{d,R}$ are unitary, all of them in generation space. Using the freedom to rotate the quark fields in generation space allows to bring the Dirac operator to the form

$$\mathbf{D}_{\mathrm{SM,q}} = \begin{pmatrix} \frac{\phi}{v}\Omega_1 C^{-1} m_u \,\Omega_2 & 0 & \slashed{D}_u + \slashed{Z} + \slashed{G} & \slashed{W}^+ \\ 0 & \frac{\phi}{v}\Omega_1 m_d \,\Omega_3 & \slashed{W}^- & \slashed{D}_d - \slashed{Z} + \slashed{G} \\ \slashed{D}_u + \slashed{G} & 0 & \frac{\phi}{v}\Omega_2^{-1} m_u C \Omega_1^{-1} & 0 \\ 0 & \slashed{D}_d + \slashed{G} & 0 & \frac{\phi}{v}\Omega_3^{-1} m_d \,\Omega_1^{-1} \end{pmatrix} \tag{4.4}$$

where

$$C = \mathcal{A}_{u,L}\mathcal{A}_{d,L}^{-1} \tag{4.5}$$

is the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and $\Omega_{1,2,3}$ are arbitrary unitary matrices (in generation space). It is manifest, by means of an appropriate choice of $\Omega_{1,2,3}$, that the physics is invariant under the redefinition $C \to U_1 C U_2$ where $U_{1,2}$ are arbitrary unitary and diagonal matrices (so that they commute with m_u and m_d). On the other hand, choosing $\Omega_{1,2,3}$ as the identity matrix simplifies the Dirac operator and shows that C has to be complex to allow violation of physical CP; the CP transformation as given in (4.2) becomes equivalent to

$$C \to C^*. \tag{4.6}$$

The similar manipulations in the lepton sector do not give rise to a matrix C since the would-be M_ν complex mass matrix vanishes. So the lepton sector does not contribute to CP violation in the effective action and this sector will be disregarded in what follows.

Alternatively, the quark rotations in generation space can be chosen so that the Dirac operator becomes

$$\mathbf{D}_{\mathrm{SM,q}} = \begin{pmatrix} \frac{\phi}{v} m_u & 0 & \slashed{D}_u + \slashed{Z} + \slashed{G} & \slashed{W}^+ C \\ 0 & \frac{\phi}{v} m_d & \slashed{W}^- C^{-1} & \slashed{D}_d - \slashed{Z} + \slashed{G} \\ \slashed{D}_u + \slashed{G} & 0 & \frac{\phi}{v} m_u & 0 \\ 0 & \slashed{D}_d + \slashed{G} & 0 & \frac{\phi}{v} m_d \end{pmatrix}. \tag{4.7}$$

This shows that only contributions involving $W^\pm$ can appear in the CP violating sector, Γ_-.

Because all quantities in the Dirac operator are c-numbers in generation space except the Yukawa couplings, and the effective action adds one-loop graphs with quarks running on them, this functional can be arranged in the form

$$\Gamma = \sum_\lambda \operatorname{tr} f_\lambda(M_u, M_d) \int d^4x \operatorname{tr} \mathcal{O}_\lambda(x) , \tag{4.8}$$

where $f_\lambda(M_u, M_d)$ are operators in generation space constructed with the complex mass matrices and $\mathcal{O}_\lambda$ are local operators constructed with the Higgs and the various gauge fields. In the first case the trace refers to generation space, $\mathcal{H}_{\text{gen}}$, and it refers to all the other spaces in the second case. As noted before all these operators do not involve any complex number in their construction in terms of the fields and mass matrices. As a consequence, the CP transformation (4.2) implies

$$\operatorname{tr} f_\lambda(M_u, M_d) \to \operatorname{tr} f_\lambda(M_u^*, M_d^*) = (\operatorname{tr} f_\lambda(M_u, M_d))^* , \tag{4.9}$$

therefore the CP violating component of the effective action can be expressed as

$$\Gamma_- = \sum_\lambda i \operatorname{Im} (\operatorname{tr} f_\lambda(M_u, M_d)) \int d^4x \operatorname{tr} \mathcal{O}_\lambda(x) . \tag{4.10}$$

It also follows that the local operators $\mathcal{O}(x)$ contributing to Γ_-^+ (normal parity, CP violating) are antihermitian, since Γ^+ is real in Euclidean space. On the other hand the local operators in Γ_-^- (abnormal parity, CP violating) must be hermitian. In this reasoning we use the fact that the two transformations involved, namely, CP (which defines the separation in (4.1)) and pseudo-parity (which defines the separation in (2.8)) commute. This is correct since pseudo-parity just exchanges the labels L and R while CP does not mix those labels.

An important remark is that in the Standard Model all bosons can be assigned natural parity, $(-1)^J$. Therefore the abnormal parity sector is just the P odd sector, while the normal parity sector of the CP odd component is C odd and P even. This is unlike the chirally broken phase of QCD where parity is preserved even by the abnormal parity sector due to the presence of abnormal parity hadrons.

5 Momentum integrals

As we shall see, for the operators $\operatorname{tr} f_\lambda(M_u, M_d)$ depending on the complex mass matrices, cf. (4.8), the derivative expansion produces the set of integrals

$$
\begin{aligned}
I_{r_1,t_1,\ldots,r_n,t_n}^k &= \\
&\int \frac{d^4p}{(2\pi)^4} (p^2)^k \operatorname{tr}\left[\frac{1}{(p^2 + M_u M_u^\dagger)^{r_1}} \frac{1}{(p^2 + M_d M_d^\dagger)^{t_1}} \cdots \frac{1}{(p^2 + M_u M_u^\dagger)^{r_n}} \frac{1}{(p^2 + M_d M_d^\dagger)^{t_n}} \right] \\
&= \int \frac{d^4p}{(2\pi)^4} (p^2)^k \operatorname{tr}\left[\frac{1}{(p^2 + m_u^2)^{r_1}} C \frac{1}{(p^2 + m_d^2)^{t_1}} C^{-1} \cdots \frac{1}{(p^2 + m_u^2)^{r_n}} C \frac{1}{(p^2 + m_d^2)^{t_n}} C^{-1} \right]
\end{aligned}
\tag{5.1}
$$

where the exponents k and r_i, t_i are non negative integers. The integral with $2n$ indices will appear in contributions with $n \, W^+$ and $n \, W^-$. (Charge conservation requires as many W^+ as W^- in any contribution to the effective action and $W^\pm$ are the only fields connecting the spaces u and d.)

Due to the cyclic property of the trace

$$I^k_{r_1,t_1,\ldots,r_n,t_n} = I^k_{r_2,t_2,\ldots,r_n,t_n,r_1,t_1}, \tag{5.2}$$

and also taking, the hermitian adjoint,

$$(I^k_{r_1,t_1,\ldots,r_n,t_n})^* = I^k_{r_n,t_{n-1},r_{n-1},\ldots,t_1,r_1,t_n}. \tag{5.3}$$

Since we have seen above that the CP violating component is tied to the imaginary part of this integral, we introduce the definition

$$\hat{I}^k_{r_1,t_1,\ldots,r_n,t_n} = i \operatorname{Im} I^k_{r_1,t_1,\ldots,r_n,t_n}, \tag{5.4}$$

which enjoys the properties

$$\hat{I}^k_{r_1,t_1,\ldots,r_n,t_n} = \hat{I}^k_{r_2,t_2,\ldots,r_n,t_n,r_1,t_1} = -\hat{I}^k_{r_n,t_{n-1},r_{n-1},\ldots,t_1,r_1,t_n}. \tag{5.5}$$

From these relations it is immediate that $\hat{I}$ vanishes if $n = 0$ or $n = 1$. Therefore, at least $2 \, C$ and $2 \, C^{-1}$ are needed to have a contribution to Γ_-, or equivalently, $2 \, W^+$ and $2 \, W^-$. This is a well known fact in the literature [8]. (This implies that the operator $\operatorname{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$ [20] mentioned in the Introduction cannot be derived from simple integration of the quarks as the would-be term with four $W^\pm$ vanishes. Such term can be produced if internal gauge field lines are allowed.)

Of particular interest will be the first non trivial case, $n = 2$. For it one finds

$$\hat{I}^k_{r_1,t_1,r_2,t_2} = -\hat{I}^k_{r_2,t_1,r_1,t_2} = -\hat{I}^k_{r_1,t_2,r_2,t_1}. \tag{5.6}$$

It will also be useful to note the transformation of the momentum integral under ud-parity (see section 3.3), namely,

$$I^k_{r_1,t_1,\ldots,r_n,t_n} \longrightarrow I^k_{t_1,\ldots,r_n,t_n,r_1}, \tag{5.7}$$

and in particular

$$\hat{I}^k_{r_1,t_1,r_2,t_2} \longrightarrow -\hat{I}^k_{t_1,r_1,t_2,r_2}. \tag{5.8}$$

Using the second form in (5.1), to compute the integrals of the type $\hat{I}^k_{r_1,t_1,r_2,t_2}$ for three generations, one can use the identity [8]

$$\operatorname{Im}(C_{ij}C^{-1}_{jk}C_{kl}C^{-1}_{li}) = J\epsilon_{ik}\epsilon_{jl}, \quad i,j,k,l = 1,2,3, \tag{5.9}$$

(with no implicit summation over repeated indices here) where $\epsilon_{ij} = \sum_{k=1}^{3} \epsilon_{ijk}$, and J is the Jarlskog invariant [8, 10]

$$J = \cos\theta_{12}\cos^2\theta_{13}\cos\theta_{23}\sin\theta_{12}\sin\theta_{13}\sin\theta_{23}\sin\delta = 3.0(2) \times 10^{-5}. \tag{5.10}$$

The resulting momentum integrals no longer involve matrices in the integrand.

The only integral required in this paper is $\hat{I}^3_{1,1,2,2}$. It can be cast in the form

$$\hat{I}^3_{1,1,2,2} = iJ\,G_F\,\kappa_{\mathrm{CP}}\,. \tag{5.11}$$

κ_{CP} is a dimensionless coefficient and G_F is the Fermi constant, which can be related to the the vacuum expectation value of the Higgs field, $G_F = 1/(\sqrt{2}v^2)$. For v and for the quark masses we take $v = 246\,\mathrm{GeV}$, $m_u = 2.55\,\mathrm{MeV}$, $m_d = 5.04\,\mathrm{MeV}$, $m_s = 104\,\mathrm{MeV}$, $m_c = 1.27\,\mathrm{GeV}$, $m_b = 4.2\,\mathrm{GeV}$, $m_t = 171.2\,\mathrm{GeV}$ [10]. This gives

$$\kappa_{\mathrm{CP}} = 3.1 \times 10^2\,, \qquad G_F\,\kappa_{\mathrm{CP}} = 3.6 \times 10^{-3}\,\mathrm{GeV}^{-2}\,. \tag{5.12}$$

The integral $\hat{I}^3_{1,1,2,2}$ is an homogeneous function of the quark masses of degree -2 so κ_{CP} can be expressed as a function of the Yukawa couplings, $y_q = \sqrt{2}m_q/v$. The loop function κ_{CP} is of great interest by itself so we give full details of its form and calculation in section 11. We only remark here that it is numerically many orders of magnitude larger than the "minimal" term $(y_u^2 - y_c^2)(y_c^2 - y_t^2)(y_t^2 - y_u^2)(y_d^2 - y_s^2)(y_s^2 - y_b^2)(y_b^2 - y_d^2) = 6 \times 10^{-18}$. The reason, of course, is that other non minimal factors appear in the full expression.

6 Derivative expansion in the SM

Smit [11] has proposed to use the derivative expansion as a suitable approach in the present context of CP violation in the SM.

In the SM each $W^\pm$, Z, $D_{u,d,\nu}$ or G_a counts as first order. As we have seen, at least four $W^\pm$ are needed in the CP violating sector, therefore Γ_- vanishes at zeroth or second order in the derivative expansion. It was shown in [11] that the fourth order is also vanishing in the abnormal parity sector. We verify below that the fourth order vanishes actually in both sectors. Therefore, as suggested in [11], the first non trivial contribution should start at six derivatives. Hernandez et al. [12] have addressed such a computation in the abnormal parity sector and find a non vanishing result. We compute below all contributions to sixth order, including Higgs and normal parity terms. Our result do not sustain those in [12]. We find non vanishing contributions in the normal parity sector but none in the abnormal parity one.

As we have noted above we do not need to consider leptons since they do not give a contribution to Γ_- assuming massless neutrinos. On the other hand we can also neglect gluons. At fourth order the four $W^\pm$ already saturate the required number of derivatives and no gluons are allowed. At sixth order there is room for up to two gluon fields. However, by gauge invariance one gluon is not admissible and two gluons must be combined to form a gluon field strength. Such term vanishes due to the trace on color. In what follows gluons are not included and color just gives a global factor $N_c = 3$.

Another question is whether the derivative expansion at low orders produces a reliable approximation to the physical amplitudes. Using simple estimates, it has been argued in [12] that the range of validity could reach the scale of the charm quark mass or even larger. Besides, it is clearly of interest to have correctly accounted for the lowest order operators of the effective action, as guidance on the available CP violating mechanisms.

7 Chiral invariant approach to the effective action

The approach of section 2.3 can be directly applied to the SM.

For the SM in the quark sector, the operator K of (2.19) takes the form (the gluons are no longer present in the covariant derivatives)

$$K = \begin{pmatrix} (\phi^2/v^2)M_u M_u^\dagger - (\slashed{D}_u + \slashed{Z})(\slashed{D}_u + \slashed{\varphi}) & -\slashed{W}^+(\slashed{D}_d + \slashed{\varphi}) \\ -\slashed{W}^-(\slashed{D}_u + \slashed{\varphi}) & (\phi^2/v^2)M_d M_d^\dagger - (\slashed{D}_d - \slashed{Z})(\slashed{D}_d + \slashed{\varphi}) \end{pmatrix}. \quad (7.1)$$

Here we have introduced the shorthand notation

$$\varphi_\mu(x) = \phi^{-1}[\partial_\mu, \phi]. \quad (7.2)$$

The operator K acts in the space $\mathcal{H}_{ud} \otimes \mathcal{H}_{\text{gen}} \otimes \mathcal{H}_{\text{color}} \otimes \mathcal{H}_{\text{Dirac}}$ (that is, the same space as $\mathbf{D}_{\text{SM,q}}$ except the factor $\mathcal{H}_{LR}$). Therefore, for the SM the equations (2.20) become

$$\Gamma^+ = -\frac{1}{2}N_c \operatorname{Re}\langle \log K \rangle,$$
$$\Gamma^- = -\frac{1}{2}N_c \, i \operatorname{Im}\langle \log K \rangle + \Gamma_{\text{gWZW}}. \quad (7.3)$$

Here we have introduced the symbol $\langle \, \rangle$ which will be used in what follows. It denotes a trace operation *including a γ_5 in the abnormal parity sector*, and just the trace, without γ_5, in the normal parity sector. The precise trace operation implied by $\langle \, \rangle$ will often be obvious from the context. The inclusion of γ_5 does not spoil the cyclic property for $\langle \, \rangle$ since all operators involved will have an even number of Dirac gamma matrices. In this way we can treat simultaneously the normal and abnormal parity components.

In (7.3) the trace implied by $\langle \, \rangle$ acts on x-space and on $\mathcal{H}_{ud} \otimes \mathcal{H}_{\text{gen}} \otimes \mathcal{H}_{\text{Dirac}}$.

It has been shown in [11] that the term Γ_{gWZW} does not have a contribution to the CP violating component of the effective action, so we disregard this term in what follows.

For convenience, let us separate K into its diagonal and off diagonal parts (in ud space)

$$K = K_D + K_A,$$
$$K_D = \begin{pmatrix} (\phi^2/v^2)M_u M_u^\dagger - (\slashed{D}_u + \slashed{Z})(\slashed{D}_u + \slashed{\varphi}) & 0 \\ 0 & (\phi^2/v^2)M_d M_d^\dagger - (\slashed{D}_d - \slashed{Z})(\slashed{D}_d + \slashed{\varphi}) \end{pmatrix},$$
$$K_A = \begin{pmatrix} 0 & -\slashed{W}^+(\slashed{D}_d + \slashed{\varphi}) \\ -\slashed{W}^-(\slashed{D}_u + \slashed{\varphi}) & 0 \end{pmatrix}. \quad (7.4)$$

The advantage of this separation is that only K_A contains charged currents, which can break CP, and also that this term is of first order in derivatives[6] (recall that the gauge fields, and in particular $W^\pm$, count as first order) so the n-th order in the derivative expansion of Γ can contain at most n factors K_A.

---

[6]K_A is of order one and not of order two because the counting refers to derivatives of the external fields and in (7.4) $\slashed{D}_{u,d}$ can still act on the quarks. As noted previously in section 2.5, such derivatives on the running fermion are of order zero in the derivative counting.

Substituting this form in $\langle \log K \rangle$, which appears in (7.3), yields

$$\langle \log K \rangle = \langle \log K_D \rangle - \frac{1}{2} \left\langle \frac{1}{K_D} K_A \frac{1}{K_D} K_A \right\rangle - \frac{1}{4} \left\langle \frac{1}{K_D} K_A \frac{1}{K_D} K_A \frac{1}{K_D} K_A \frac{1}{K_D} K_A \right\rangle - \cdots$$

$$= \sum_{n=0}^{\infty} \langle \log K \rangle_{2n}. \tag{7.5}$$

The subindex $2n$ in $\langle \log K \rangle_{2n}$ indicates that the term contains exactly $2n$ $W^{\pm}$ fields. Working out the trace in ud space (for $n > 0$) one obtains

$$\langle \log K \rangle_{2n} = -\frac{1}{n} \left\langle \left[\left(\frac{\phi^2}{v^2} M_u M_u^\dagger - (\slashed{D}_u + \slashed{Z})(\slashed{D}_u + \slashed{\varphi}) \right)^{-1} \slashed{W}^+ (\slashed{D}_d + \slashed{\varphi}) \right. \right.$$

$$\left. \left. \times \left(\frac{\phi^2}{v^2} M_d M_d^\dagger - (\slashed{D}_d - \slashed{Z})(\slashed{D}_d + \slashed{\varphi}) \right)^{-1} \slashed{W}^- (\slashed{D}_u + \slashed{\varphi}) \right]^n \right\rangle. \tag{7.6}$$

Here the trace implied by $\langle \ \rangle$ refers to $\mathcal{H}_{\text{gen}} \otimes \mathcal{H}_{\text{Dirac}}$, and x-space. This expression is manifestly ud-parity invariant (see section 3.3), as can be verified by using the trace cyclic property.

Because at least four $W^{\pm}$ are required to have a CP violating term, the relevant contributions start at $\langle \log K \rangle_4$. To sixth order in the derivative expansion only $\langle \log K \rangle_4$ and $\langle \log K \rangle_6$ have to be retained. $\langle \log K \rangle_4$ contains terms with at least four derivatives (namely, those coming from $W^{\pm}$), while $\langle \log K \rangle_6$ starts at six derivatives:

$$\langle \log K \rangle_4 = \langle \log K \rangle_{4+0} + \langle \log K \rangle_{4+2} + \mathcal{O}(D^8),$$
$$\langle \log K \rangle_6 = \langle \log K \rangle_{6+0} + \mathcal{O}(D^8). \tag{7.7}$$

Here $\langle \log K \rangle_{2n+2m}$ indicates $2n$ derivatives from $W^{\pm}$ and $2m$ more derivatives not from $W^{\pm}$, i.e., coming from ∂_μ, Z_μ or $A_\mu^{u,d}$.

8 Vanishing of terms of the type $2n + 0$

In ref. [11], and also confirmed in [12], it was shown that there is no CP violating contribution to four derivatives in the abnormal parity sector. Let us show that this is true in both sectors and that, moreover, there is no sixth order contribution either coming from terms with six $W^{\pm}$.

These statements are remarkably easy to establish using the method of symbols described in section 2.5. This method amounts to make the replacement $\slashed{D}_{u,d} \to \slashed{D}_{u,d} + \slashed{p}$ in K_D and K_A and integrate over p_μ. (Recall that p_μ is purely imaginary but $p^2 = -p_\mu^2$, as explained in section 2.5.)

Concretely, the CP violating terms with precisely four derivatives must come from $\langle \log K \rangle_4$ taking no other derivatives than the four $W^{\pm}$ (i.e., must be of the type $4 + 0$). Therefore, to four derivatives, we can set $\slashed{D}_{u,d} \to \slashed{p}$, $\slashed{Z} \to 0$ and $\slashed{\varphi} \to 0$ in the operator K:

$$\langle \log K \rangle_{4+0} = -\frac{1}{2} \int \frac{d^4x\, d^4p}{(2\pi)^4} \left\langle \left[\left(\frac{\phi^2}{v^2} M_u M_u^\dagger + p^2 \right)^{-1} \slashed{W}^+ \slashed{p} \left(\frac{\phi^2}{v^2} M_d M_d^\dagger + p^2 \right)^{-1} \slashed{W}^- \slashed{p} \right]^2 \right\rangle. \tag{8.1}$$

Here it is already obvious that, upon momentum integration, the integral $I^2_{1,1,1,1}$ (introduced in (5.1)) will be generated. Because this integral is real, due to eq. (5.3), it follows (cf. (4.10)) that no CP violating term is produced to fourth order in the derivative expansion, neither in the normal nor the abnormal parity sectors.

To see this in more detail, we first take an angular average in (8.1), using

$$p_\mu p_\nu p_\alpha p_\beta \to (\delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\alpha\nu})p^4/(d(d+2)). \tag{8.2}$$

Since no derivatives with respect to x are present in the expression, we can simply rescale $p_\mu \to (\phi(x)/v)p_\mu$, and the momentum integrals in (5.1) apply. Specifically,

$$\langle \log K \rangle_{4+0} = -\frac{1}{48}I^2_{1,1,1,1} \int d^4x \, \big\langle \left(W^+\gamma_\mu W^-\gamma_\mu \right)^2 + \left(W^+\gamma_\mu W^-\gamma_\nu \right)^2$$
$$+ W^+\gamma_\mu W^-\gamma_\nu W^+\gamma_\nu W^-\gamma_\mu \big\rangle. \tag{8.3}$$

The result to four derivatives is proportional to $I^2_{1,1,1,1}$, as advertised.

If we consider now the case of sixth order $6 + 0$, i.e., when the six derivatives are saturated by six $W^\pm$, it is quite clear, by using the same reasoning, that the result will be proportional to $I^2_{1,1,1,1,1,1}$, which is also real, and therefore, also no CP violation is produced in either sector from such contributions. Of course, the analogous result holds for all orders of the type $2n + 0$ too. That is,

$$(\Gamma_-)_{2n+0} = 0. \tag{8.4}$$

We note that this vanishing is rather trivial in the abnormal parity sector (for space-time dimension $d > 2$): with the Levi-Civita tensor and only two four-vectors $W^\pm_\mu$ it is not possible to construct a non vanishing scalar.

The vanishing for the normal parity part is also easily understood. Due to charge conservation, the possible operators constructed using only $W^\pm_\mu$ are of the type $((W^+_\mu W^+_\mu)(W^-_\nu W^-_\nu))^n (W^+_\alpha W^-_\alpha)^m$ and are CP even.

9 CP violation in the absence of Higgs field derivatives

We have just seen that no CP violation occurs to four derivatives and also to six derivatives if these are saturated by $W^\pm$'s. Therefore, any CP violation through sixth order in the derivative expansion must come from terms with four $W^\pm$ and two other derivatives not of the $W^\pm$ type, that is, terms of the type $4 + 2$:

$$\Gamma_- = (\Gamma_-)_{4+2} + \mathcal{O}(D^8). \tag{9.1}$$

In the next section we shall consider the general case. Presently, we study the simplest situation where no derivatives of the Higgs field are considered. In this case, the Higgs field $\phi(x)$ itself can be set to its vacuum expectation value v, since it can be restored in the formulas at the end by a rescaling of the quark masses. Under these assumptions, the trace $\langle \log K \rangle_4$ of (7.6) reduces to the simpler form

$$\langle \log K \rangle_4 = -\frac{1}{2}\Big\langle \Big[\left(M_u M_u^\dagger - (\slashed{D}_u + \slashed{Z})\slashed{D}_u \right)^{-1} W^+ \slashed{D}_d \left(M_d M_d^\dagger - (\slashed{D}_d - \slashed{Z})\slashed{D}_d \right)^{-1} W^- \slashed{D}_u \Big]^2 \Big\rangle. \tag{9.2}$$

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Here we apply once again the method of symbols making the replacement $D_\mu \to p_\mu + D_\mu$, integrating over p_μ and then expanding in powers of the derivatives:

$$\langle \log K \rangle_4 = -\frac{1}{2} \int \frac{d^4x d^4p}{(2\pi)^4} \Big\langle \tilde{N}_u \rlap{/}{W}^+(\rlap{/}{p} + \rlap{/}{D}_d) \tilde{N}_d \rlap{/}{W}^-(\rlap{/}{p} + \rlap{/}{D}_u) \tilde{N}_u \rlap{/}{W}^+(\rlap{/}{p} + \rlap{/}{D}_d) \tilde{N}_d \rlap{/}{W}^-(\rlap{/}{p} + \rlap{/}{D}_u) \Big\rangle. \tag{9.3}$$

Where

$$\tilde{N}_u = \big(M_u M_u^\dagger - (\rlap{/}{p} + \rlap{/}{D}_u + \rlap{/}{Z})(\rlap{/}{p} + \rlap{/}{D}_u)\big)^{-1}, \quad \tilde{N}_d = \big(M_d M_d^\dagger - (\rlap{/}{p} + \rlap{/}{D}_d - \rlap{/}{Z})(\rlap{/}{p} + \rlap{/}{D}_d)\big)^{-1}. \tag{9.4}$$

Expansion of the first denominator gives

$$\tilde{N}_u = N_u + N_u^2\big(\rlap{/}{p}\,\rlap{/}{D}_u + (\rlap{/}{D}_u + \rlap{/}{Z})\rlap{/}{p} + (\rlap{/}{D}_u + \rlap{/}{Z})\rlap{/}{D}_u\big) + N_u^3\big(\rlap{/}{p}\,\rlap{/}{D}_u + (\rlap{/}{D}_u + \rlap{/}{Z})\rlap{/}{p}\big)^2 + \mathcal{O}(D^3). \tag{9.5}$$

and similarly for $\tilde{N}_d$. And we have defined

$$N_u = \big(M_u M_u^\dagger + p^2\big)^{-1}, \quad N_d = \big(M_d M_d^\dagger + p^2\big)^{-1}. \tag{9.6}$$

The two objects N_u and N_d are x-independent, they do not commute with each other but commute with all other quantities in $\langle \log K \rangle_4$. In addition, they appear in the momentum integrals introduced in section 5. Indeed, the first eq. in (5.1) can be rewritten as

$$I_{r_1,t_1,\ldots,r_n,t_n}^k = \int \frac{d^4p}{(2\pi)^4} (p^2)^k \, \mathrm{tr}\big[N_u^{r_1} N_d^{t_1} \cdots N_u^{r_n} N_d^{t_n}\big]. \tag{9.7}$$

At this point considerable simplification can be achieved by making the following observation. To produce $\langle \log K \rangle_{4+2}$ from (9.3) we need to pick up exactly two derivatives (apart from the four explicit $W^\pm$). On the other hand, CP violating contributions come only from the imaginary part of I_{r_1,t_1,r_2,t_2}^k, and this requires $r_1 \neq r_2$ and $t_1 \neq t_2$, cf. (5.6). Now, it is quite clear that, in order to obtain such a situation, it is necessary to pick up exactly one of the derivatives from one of the $\tilde{N}_u$ and the other derivative from one of the $\tilde{N}_d$. Any other possibility ends up with either the two N_u or the two N_d raised to the same power, i.e., $r_1 = r_2 = 1$ or $t_1 = t_2 = 1$. Therefore the CP violating contributions in the present case come from $I_{1,1,2,2}^k$.

To work this out let us simplify the expressions by introducing the following quantities, which appear naturally in (9.3) and (9.5)

$$w^\pm = \rlap{/}{W}^\pm \rlap{/}{p}, \quad \delta_u = 2pD_u + \rlap{/}{Z}\,\rlap{/}{p}, \quad \delta_d = 2pD_d - \rlap{/}{Z}\,\rlap{/}{p}. \tag{9.8}$$

Then, applying the previous observation yields

$$\begin{aligned}
\langle \log K \rangle_{4+2} = &-\frac{1}{2} \int \frac{d^4x d^4p}{(2\pi)^4} \Big\langle N_u^2 \delta_u w^+ N_d^2 \delta_d w^- N_u w^+ N_d w^- + N_u^2 \delta_u w^+ N_d w^- N_u w^+ N_d^2 \delta_d w^- \\
&+ N_u w^+ N_d^2 \delta_d w^- N_u^2 \delta_u w^+ N_d w^- + N_u w^+ N_d w^- N_u^2 \delta_u w^+ N_d^2 \delta_d w^- \Big\rangle \\
&+ \text{CP invariant terms} \\
= &-\frac{1}{2} \int \frac{d^4x d^4p}{(2\pi)^4} \mathrm{tr}\,(N_u N_d N_u^2 N_d^2) \\
&\times \Big\langle \delta_u w^+ \delta_d w^- w^+ w^- - \delta_u w^+ w^- w^+ \delta_d w^- - w^+ \delta_d w^- \delta_u w^+ w^- \\
&+ w^+ w^- \delta_u w^+ \delta_d w^- \Big\rangle + \text{CP i.t.}
\end{aligned} \tag{9.9}$$

In the second equality we have rearranged the factors N_u, N_u^2, N_d, N_d^2, using that the CP violating part of the momentum integral, $\hat{I}^k_{r_1,t_1,r_2,t_2}$, is antisymmetric under exchange of r_1,r_2 or t_1,t_2.

The integrand in (9.9) contains derivatives (inside $\delta_{u,d}$) which are not derivating anything yet. As explained in section 2.5, in general one proceeds by moving the derivatives to the right, and at the end the momentum integral kills these "free" derivatives. In the present case this turns out not to be necessary. Instead, we can introduce the combinations

$$(\delta w)^+ = \delta_u w^+ - w^+ \delta_d, \qquad (\delta w)^- = \delta_d w^- - w^- \delta_u, \tag{9.10}$$

in such a way that

$$\langle \log K \rangle_{4+2} = -\frac{1}{2} \int \frac{d^4x\, d^4p}{(2\pi)^4} \mathrm{tr}\, (N_u N_d N_u^2 N_d^2) \left\langle (\delta w)^+ w^- (\delta w)^+ w^- - w^+ (\delta w)^- w^+ (\delta w)^- \right\rangle$$
$$+ \text{ CP i.t.} \tag{9.11}$$

and the integrand no longer contains any "free" derivative.

The CP violating part of this result is proportional to $\hat{I}^3_{1,1,2,2}$ and develops a factor $(v/\phi(x))^2$ upon restoration of the Higgs.

There is a simple observation that can already be made at the present stage. Namely, *there is no CP violating contribution to the abnormal parity sector* from terms of the type $4 + 2$ without Higgs field derivatives. As discussed in section 4, in the abnormal parity sector the operator multiplying the momentum integral must be *hermitian* to have a CP violating contribution, however,

$$\mathcal{O} = \mathrm{tr}\, \left[\gamma_5 \left((\delta w)^+ w^- (\delta w)^+ w^- - w^+ (\delta w)^- w^+ (\delta w)^- \right) \right] \tag{9.12}$$

is purely imaginary. To verify this, we take the complex conjugate of everything inside $\mathcal{O}$. In Euclidean four-dimensional space γ_μ and γ_5 are related to their complex conjugates by a common similarity transformation,

$$\gamma_\mu^* = C_c^{-1} \gamma_\mu C_c, \quad \gamma_5^* = C_c^{-1} \gamma_5 C_c. \tag{9.13}$$

Also, one verifies that

$$(w^\pm)^* = C_c^{-1} w^\mp C_c, \quad ((\delta w)^\pm)^* = -C_c^{-1} (\delta w)^\mp C_c. \tag{9.14}$$

Therefore

$$\mathcal{O}^* = -\mathcal{O}, \tag{9.15}$$

as advertised.

In the absence of $F_{\mu\nu}^{u,d}$ and of complex quark mass matrices, complex conjugation becomes equivalent to ud-parity.[7] So $\mathcal{O}$ is imaginary (in the normal and in the abnormal parity sectors) because it is odd under ud-parity. In turn this was obvious without further

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[7]Under complex conjugation $W_\mu^\pm \to -W_\mu^\mp$, $Z_\mu \to -Z_\mu$ (in Euclidean space) while under ud-parity $W_\mu^\pm \to W_\mu^\mp$, $Z_\mu \to -Z_\mu$.

calculation once the momentum integral $\hat{I}^3_{1,1,2,2}$ was obtained in (9.9). This is because the latter is odd under ud-parity and the full effective action is even (cf. section 3.3).

Remarkably, the operator $\text{tr}\,[\gamma_5(\delta w)^+ w^-(\delta w)^+ w^-]$ (and hence its complex conjugate) vanishes by itself after taking an angular average and the Dirac trace. We have not found a simple explanation for this.

The operator $\mathcal{O}$ in the normal parity sector (i.e., as in (9.12) without γ_5) is also odd under ud-parity and so also purely imaginary. Therefore the operation of taking the real part indicated in (7.3) is redundant. The normal parity contribution is not vanishing. The result so obtained is part of the general result which we present in the next section.

10   CP violating terms to six derivatives

In this section we present the full result for the CP violating terms of the effective action to six derivatives. This includes all relevant fields in the SM, and derivatives of the Higgs field.

We have used the method of symbols and repeated the calculation using the method of covariant symbols as a check, to obtain precisely the same result from both calculations. In the latter case we use the covariant derivatives $D_{u,\mu}$ and $D_{d,\mu}$ to carry out the construction indicated in (2.26). This full result is also consistent with the independent computation made in the previous section.

From the calculation we obtain the remarkable result that $\langle \log K \rangle_{4+2}$ vanishes identically in the abnormal parity sector for all terms that could have a contribution to CP violation. In fact, Γ^- vanishes for all the terms that we have computed, whether CP violating or not. (Of course we have not studied most CP invariant terms with six or less derivatives, as they are not required for our purposes.) We have not found a compelling reason for this, so most likely, the vanishing found is just a low order accidental symmetry. The existence of CP violating terms in the abnormal parity sector of the SM with eight derivatives or more is not excluded. These would be the leading P violating contributions, relevant to the electric dipole moment problem.

Another unexpected result is that $\langle \log K \rangle_{4+2}$ is purely real in the normal parity sector, for terms that contribute to CP violation. (And so, taking the real part indicated in (7.3) becomes redundant.) In the calculation this follows from the fact that only the momentum integral $\hat{I}^3_{1,1,2,2}$ appears. This integral is odd under ud-parity and hence the accompanying operator must be odd too and hence imaginary. Once again it not obvious to us whether this feature will be maintained at higher orders in the derivative expansion, coming from an exact selection rule in the SM, or is just an accidental symmetry.

The result, in Minkowski space and in the unitary gauge, reads

$$\Gamma_{\text{SM}} = -\frac{N_c}{2} i J \, G_F \, \kappa_{\text{CP}} \int d^4x \left(\frac{v}{\phi}\right)^2 (\mathcal{O}_0 + \mathcal{O}_1 + \mathcal{O}_2) + \mathcal{O}(D^8) + \text{CP invariant terms}\,. \quad (10.1)$$

Here $N_c = 3$ is the number of colors, J the Jarlskog invariant, G_F the Fermi constant and $\kappa_{\text{CP}} = 3.1 \times 10^2$ is the dimensionless parameter of section 5. The operators $\mathcal{O}_i$, $i = 0, 1, 2$, have dimension six and are all purely imaginary:

$$\mathcal{O}_0 = \frac{2}{3}W_\mu^+ W_{\mu\nu}^- W_\alpha^+ W_{\nu\alpha}^- - \frac{2}{3}W_\mu^+ W_{\mu\nu}^- W_\alpha^+ W_{\alpha\nu}^-$$
$$+\frac{4}{3}W_\mu^+ W_{\nu\mu}^- W_\nu^+ W_{\alpha\alpha}^- - 2W_\mu^+ W_{\nu\mu}^- W_\alpha^+ W_{\nu\alpha}^-$$
$$+\frac{2}{3}W_\mu^+ W_{\nu\mu}^- W_\alpha^+ W_{\alpha\nu}^- - \frac{1}{3}W_\mu^+ W_{\nu\nu}^- W_\mu^+ W_{\alpha\alpha}^-$$
$$+\frac{5}{3}W_\mu^+ W_{\nu\alpha}^- W_\mu^+ W_{\nu\alpha}^- - \frac{1}{3}W_\mu^+ W_{\nu\alpha}^- W_\mu^+ W_{\alpha\nu}^- - \text{c.c.}$$

$$\mathcal{O}_1 = \frac{8}{3}(\varphi_\mu + iZ_\mu)$$
$$\times \Big(W_\nu^+ W_\mu^- W_\nu^+ W_{\alpha\alpha}^- - W_\nu^+ W_\nu^- W_\mu^+ W_{\alpha\alpha}^-$$
$$+ W_\nu^+ W_\alpha^- W_\mu^+ W_{\alpha\nu}^- - W_\nu^+ W_\alpha^- W_\nu^+ W_{\mu\alpha}^-$$
$$- W_\nu^+ W_\alpha^- W_\nu^+ W_{\alpha\mu}^- + W_\nu^+ W_\alpha^- W_\alpha^+ W_{\mu\nu}^- \Big) - \text{c.c.}$$

$$\mathcal{O}_2 = -\frac{4}{3}(\varphi_\mu + iZ_\mu)(\varphi_\mu + iZ_\mu)\Big(W_\nu^+ W_\nu^- W_\alpha^+ W_\alpha^- - 2W_\nu^+ W_\alpha^- W_\nu^+ W_\alpha^- \Big)$$
$$-\frac{4}{3}(\varphi_\mu + iZ_\mu)(\varphi_\nu + iZ_\nu) \qquad\qquad (10.2)$$
$$\times \Big(W_\mu^+ W_\alpha^- W_\nu^+ W_\alpha^- - 2W_\alpha^+ W_\mu^- W_\alpha^+ W_\nu^- + 2W_\alpha^+ W_\nu^- W_\mu^+ W_\alpha^- \Big) - \text{c.c.}$$

In these expressions c.c stands for complex conjugate. Even if these expressions refer to Minkowski space, the Lorentz indices are all written as subindices for clarity as no ambiguity may arise. With the conventions given above, the pass from Euclidean to Minkowskian metric amounts to the replacement $Z_\mu \to iZ_\mu$ with no other change.

As noted in section 2.4, the effective action in Minkowski space is purely real, at every order in the derivative expansion and this property is found here.

The fields $W_{\mu\nu}^\pm$ were defined in section 3.1 and they are expressed in terms of the $U_{\text{e.m.}}(1)$ covariant derivative in (3.17). On the other hand φ_μ was defined in (7.2) as the logarithmic derivative of the Higgs field.

As it is readily verified, all the operators $\mathcal{O}_{0,1,2}$ are indeed odd under the CP transformation

$$\varphi_\mu(x) \pm iZ_\mu(x) \to \pi_\mu{}^\nu(\varphi_\nu(\tilde{x}) \mp iZ_\nu(\tilde{x})),$$
$$W_\mu^\pm(x) \to -\pi_\mu{}^\nu W_\nu^\mp(\tilde{x}), \qquad\qquad W_{\mu\nu}^\pm(x) \to -\pi_\mu{}^\alpha \pi_\nu{}^\beta W_{\alpha\beta}^\mp(\tilde{x}). \qquad (10.3)$$

Terms including the field strengths $F_{\mu\nu}^{u,d}$ are absent. This can be understood from the fact that the available operators, $iF_{\mu\nu}^{u,d}W_\mu^+ W_\nu^- W_\alpha^+ W_\alpha^-$, are CP even.

On the other hand, the result in (10.2) presents some regularities which for us remain purely "empirical". In $\mathcal{O}_0$ terms coupling $W_{\mu\nu}^+$ to $W_{\alpha\beta}^-$ do not appear. It is always possible to change variables from φ_μ and Z_μ to $\varphi_\mu \pm iZ_\mu$, however, it is not obvious why, in $\mathcal{O}_1$, the combinations $\varphi_\mu \pm iZ_\mu$ couple only to $W_{\alpha\beta}^\mp$ and not to $W_{\alpha\beta}^\pm$. Also it is not clear why in $\mathcal{O}_2$, the combinations $\varphi_\mu \pm iZ_\mu$ couple only with themselves and not with $\varphi_\mu \mp iZ_\mu$. This latter observation suggests the speculation that the effective action (or perhaps Γ_-^+) to all orders

could be of the form $F[\varphi_\mu + iZ_\mu] - F[\varphi_\mu - iZ_\mu]$, where the functional F would depend analytically (holomorphically) on its argument.

Also, the relatively simple dependence of the result on the combinations $\varphi_\mu \pm iZ_\mu$ suggests the possibility of reconstructing the full result with Higgs derivatives from that without φ_μ (by some kind of gauging). In this case the calculation in section 9 could perhaps be adapted to include φ_μ. We have not tried this in this work.

The fields $\varphi_\mu - iZ_\mu$ and iW_μ^+ follow from projection of $\nabla_\mu \Phi$ onto Φ and $\tilde{\Phi}$, respectively, where ∇_μ represents the full $\mathrm{SU}_L(2) \times \mathrm{U}_Y(1)$ covariant derivative, and $W_{\mu\nu}^+$ can also be written using Φ and $\tilde{\Phi}$ and their covariant derivatives, but the result is not particularly illuminating.

11 The coefficient κ_{CP}

In this section we study in some depth the function κ_{CP}. Using the second form in (5.1) as well as the identity (5.9), the momentum integral $\hat{I}_{1,1,2,2}^3$ takes the form

$$\hat{I}_{1,1,2,2}^3 = iJ \sum_{i,j,k,l=1}^{3} \epsilon_{ik}\epsilon_{jl} \int \frac{d^4p}{(2\pi)^4} (p^2)^3 \frac{1}{(p^2+m_{u,i}^2)} \frac{1}{(p^2+m_{d,j}^2)} \frac{1}{(p^2+m_{u,k}^2)^2} \frac{1}{(p^2+m_{d,l}^2)^2}.$$
(11.1)

Here $m_{u,i}$ denotes the mass of the quark of type u of the i-th generation, and similarly for $m_{d,i}$.

The sum over flavors is easily carried out using the identity

$$\sum_{i,k}^{3} \epsilon_{ik} \frac{1}{(p^2+m_{u,i}^2)} \frac{1}{(p^2+m_{u,k}^2)^2} = -\frac{(m_u^2-m_c^2)(m_c^2-m_t^2)(m_t^2-m_u^2)}{(p^2+m_u^2)^2(p^2+m_c^2)^2(p^2+m_t^2)^2},$$
(11.2)

and similarly for d-type quarks. The integral can then be written as

$$\hat{I}_{1,1,2,2}^3 = iJ\,\Delta_m\,I_m.$$
(11.3)

where

$$\Delta_m = (m_u^2-m_c^2)(m_c^2-m_t^2)(m_t^2-m_u^2)(m_d^2-m_s^2)(m_s^2-m_b^2)(m_b^2-m_d^2),$$
(11.4)

$$I_m = \int \frac{d^4p}{(2\pi)^4} (p^2)^3 \prod_{q=1}^{6} \frac{1}{(p^2+m_q^2)^2}.$$
(11.5)

The factor $J\,\Delta_m$ is just the Jarlskog determinant Δ of (1.1). This is the "minimal" factor that must always be present in the CP odd effective action as dictated by perturbation theory. This can be seen for instance considering the same integral at finite temperature which amounts to replace the energy integral by a fermionic Matsubara sum. In the limit of sufficiently high temperature all quark masses can be treated as perturbations. So the known perturbative result applies and it starts at order twelve with $J\,\Delta_m$. As a consequence the integral $\hat{I}_{1,1,2,2}^3$ is consistent with the well known fact that no CP breaking can take place if two up-type or two down-type quark have the same finite mass. (The case of degeneracy with vanishing mass requires a separate study since I_m could present infrared divergences.)

In I_m the product is over the six quark flavors, regardless of its u or d type. So the coefficient $\kappa_{\rm CP}$ introduced in section 5 has the same symmetry as Δ_m under exchange of quark flavors, the factor I_m being completely symmetric, and actually positive definite.

The momentum integrals of the type I_m, typical of zero momentum insertions in a Feynman graph, can be computed using the relation [17]

$$\int \frac{d^d p}{(2\pi)^d} (p^2)^k \prod_{j=1}^n \frac{1}{(p^2 + m_j^2)^{r_j}} = \frac{(-1)^{k+d/2-1+\sum_j r_j}}{(4\pi)^{d/2}\Gamma(d/2)} \mathcal{I}_{r_1,\dots,r_n}^{k+d/2-1}, \qquad k + d/2 = 1, 2, 3, \dots \tag{11.6}$$

where

$$\mathcal{I}_{r_1,\dots,r_n}^\alpha(m_1,\dots,m_n) = \oint \frac{dz}{2\pi i} z^\alpha \log(z) \prod_{j=1}^n \frac{1}{(z - m_j^2)^{r_j}}, \tag{11.7}$$

and the integration is along a positive closed simple contour enclosing the poles at m_j^2 but excluding $z = 0$, and the cut is on the real negative axis. The identity (11.6) assumes positive m_j^2 and holds whenever the left-hand side is ultraviolet and infrared finite. If it is not, the right-hand side gives the finite part.[8]

Using (11.6), and comparing with the definition of the loop function $\kappa_{\rm CP}$ in (5.11) yields

$$\kappa_{\rm CP} = 2^{3/2} \Delta_y I_y, \tag{11.8}$$

where

$$\Delta_y = (y_u^2 - y_c^2)(y_c^2 - y_t^2)(y_t^2 - y_u^2)(y_d^2 - y_s^2)(y_s^2 - y_b^2)(y_b^2 - y_d^2) = (\sqrt{2}/v)^{12}\,\Delta_m,$$
$$I_y = \frac{1}{(4\pi)^2} \mathcal{I}_{2,2,2,2,2,2}^4(y_u, y_c, y_t, y_d, y_s, y_b) = (v/\sqrt{2})^{14}\, I_m\,, \tag{11.9}$$

and the Yukawa couplings $y_q = \sqrt{2} m_q/v$ are used. The coupling controlling the strength of the CP violating operators has dimensions of one over mass squared. The use of the Yukawa coupling amounts to using v as convenient mass scale to measure this coupling.

The contour integrals (11.7) are readily computed by residues.[9] This produces the explicit expression

$$I_y = \frac{1}{(4\pi)^2} \sum_i \left[\frac{y_i^6(1 + 4\log y_i^2)}{\prod_j'(y_i^2 - y_j^2)^2} - \frac{y_i^8 \log y_i^2}{\prod_j'(y_i^2 - y_j^2)^3}\left(10 y_i^8 - 8 y_i^6 \sum_j{}' y_j^2 \right.\right.$$
$$\left.\left. + 6 y_i^4 \sum_{j<k}{}' y_j^2 y_k^2 - 4 y_i^2 \sum_{j<k<l}{}' y_j^2 y_k^2 y_l^2 + 2\sum_{j<k<l<r}{}' y_j^2 y_k^2 y_l^2 y_r^2 \right)\right]. \tag{11.10}$$

In this expression the indices i, j, k, l, r run over the six quark flavors. The prime in a sum or product indicates to omit the term i in that sum or product. Although the explicit

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[8]Note that when α is a non negative integer, the contour integral scales as $1 + \alpha - \sum_j r_j$ but in the presence of infrared divergencies an anomalous scale term develops from the logarithm .

[9]

$$\mathcal{I}_{r_1,\dots,r_n}^\alpha(m_1,\dots,m_n) = \sum_{i=1}^n \frac{1}{(r_i - 1)!} \frac{d^{r_i-1}}{d(m_i^2)^{r_i-1}} \frac{(m_i^2)^\alpha \log(m_i^2)}{\prod_{j=1,j\neq i}^n (m_i^2 - m_j^2)^{r_j}}\,.$$

expression of the integral I_y looks divergent as two masses become degenerated this is not so, as is obvious from the integral itself: in the coincidence limit the integral (11.5) is perfectly regular. Also, I_y yields an homogeneous function of the y of degree -14. The inhomogeneous scale variation from the logarithms cancels after adding the six terms. (This implies that one can use one of the Yukawa couplings, y_q, as overall scale, $y_i \rightarrow y_i/y_q$ to remove one of the six logarithms in the expression, at the price of a less symmetric formula.) The formula can be written in many different ways but none is expected to give a simple expression. The simplest and most transparent form is perhaps its very definition as a momentum integral, (11.5).

Each of the factors Δ_y, I_y in κ_{CP} gives quite disparate numbers, namely, $\Delta_y = 6.0 \times 10^{-18}$ and $I_y = 1.8 \times 10^{19}$. This is because they are homogeneous functions of very large degree (high mass dimension) and so they change wildly under even moderate changes in the scale. For instance, in units of the bottom quark mass $\Delta_m = 1.5 \times 10^2$ and $I_m = 4.1 \times 10^{-4}$. Their product has degree -2 and so the number is much less dependent on the mass scale.

Therefore the huge cancellation between factors is partially trivial, and as noted in [11] the factor Δ_y should not be used as a rough estimate of the CP violating component. What is not trivial is the detailed role played by the light quarks. The value of κ_{CP} is enhanced by the small mass of the quarks u, d and s. This can be seen by changing artificially their mass, $m_u \rightarrow \lambda m_u$, $m_d \rightarrow \lambda m_d$, $m_s \rightarrow \lambda m_s$. As λ moves in the range from 1 to 25 one finds that I_y decreases monotonously by a factor 5000. This rather large factor corresponds to an effective dimension which runs from about -2 at $\lambda = 1$ to about -2.5 at the highest value of λ. The value -2 corresponds to the scale dimension at the infrared divergent point $\lambda = 0$. On the other hand the factor Δ_y increases by a factor about 400. Beyond this range Δ_y reaches a maximum and starts to fall heading for the zero at $\lambda = m_b/m_s \sim 40$, and then starts to grow again with opposite sign. The net effect is to find a quenching of κ_{CP} as one departs from the SU(3) chiral point. Nevertheless, it should be noted that κ_{CP} is a rather complicated function of the quark masses, without well defined sign, so any interpretation should be taken with some caution as the result might look different depending on how the masses are moved in detail.

The coefficient κ_{CP} is infrared finite as any two quark masses go to zero while the other quarks remain massive. This allows us to consider the limiting case $m_u = m_d = 0$. (The coefficient vanishes trivially if the two massless quarks are u-like or d-like, due to the antisymmetry of the Δ_y factor.) This gives for the I_y factor

$$I_y = \frac{1}{(4\pi)^2} \mathcal{I}^0_{2,2,2,2}(y_c, y_t, y_s, y_b) \tag{11.11}$$
$$= \frac{1}{(4\pi)^2} \sum_i \left[\frac{1}{y_i^2} \frac{1}{\prod_j'(y_i^2 - y_j^2)^2} + \frac{\log y_i^2}{\prod_j'(y_i^2 - y_j^2)^3} \left(-6y_i^4 + 4y_i^2 \sum_j' y_j^2 - 2\sum_{j<k}' y_j^2 y_k^2 \right) \right].$$

Here the indices i, j, k run over the four remaining flavors s, c, b, t.

The approximation $m_u, m_d \rightarrow 0$ does not change much the value of the coefficient κ_{CP}, overestimating it by a 3% as compared to the exact value. Since the mass of the quark s is also rather small one can consider the further limit $m_s \rightarrow 0$. The limit of vanishing

m_u, m_d, m_s is affected by infrared divergencies and this manifests in the fact that while $\kappa_{\rm CP}$ remains finite, the value depends on how this limit is taken. Explicitly, the dominant term as m_u, m_d, m_s become small is

$$\Delta_y I_y = \frac{1}{(4\pi)^2} \left[\frac{-y_d^4 y_s^4 (y_d^2 + y_s^2) + 2 y_u^2 y_d^2 y_s^2 (y_d^4 + y_s^4) - y_u^4 (y_d^6 + y_s^6) - y_u^6 (y_d^2 - y_s^2)^2}{(y_d^2 - y_s^2)(y_u^2 - y_d^2)^2 (y_u^2 - y_s^2)^2} \right.$$
$$- 2 y_d^6 \log(y_d^2/y_u^2) \frac{2 y_u^2 y_s^2 - y_u^2 y_d^2 - y_d^2 y_s^2}{(y_d^2 - y_s^2)^2 (y_d^2 - y_u^2)^3} \tag{11.12}$$
$$\left. + 2 y_s^6 \log(y_s^2/y_u^2) \frac{2 y_u^2 y_d^2 - y_u^2 y_s^2 - y_d^2 y_s^2}{(y_d^2 - y_s^2)^2 (y_s^2 - y_u^2)^3} \right] \left(\frac{1}{y_c^2} - \frac{1}{y_t^2} \right) + O(y_{\rm light\ quark}^2).$$

The terms retained are those which are homogenous of degree zero in y_u, y_d, y_s, while the reminder has degree 2. This function is antisymmetric under the exchange of d and s. A quite remarkable fact is that, to this order, the dependence on y_c and y_t factorizes and moreover y_b does not appear in the expression.

The above function is finite but far from continuous in the massless limit. The physical situation is $m_u, m_d \ll m_s$ so a sensible limit to consider corresponds to taking first $m_u, m_d \to 0$ and later $m_s \to 0$. In this approximation

$$\Delta_y I_y = \frac{1}{(4\pi)^2} \left(\frac{1}{y_c^2} - \frac{1}{y_t^2} \right). \tag{11.13}$$

Equivalently, in this SU(3) chiral limit

$$G_F \, \kappa_{\rm CP} = \frac{1}{(4\pi)^2} \left(\frac{1}{m_c^2} - \frac{1}{m_t^2} \right). \tag{11.14}$$

The full result corresponds rather to $0.92/((4\pi)^2 m_c^2)$ so this approximation overestimates it by an 8%. Despite the simplicity of this approximation the massless SU(3) chiral limit is highly non trivial as can be seen from (11.12), and in fact even the sign is not obvious from the expression.

12 Conclusions

We have studied the CP breaking terms in the effective action of the Standard Model obtained by integration of quarks and leptons, including operators up to dimension six. The result of the calculation is summarized in eqs. (10.1) and (10.2).

One of the main results is that such CP breaking operators appear with a sizable coupling, of the order of 5×10^2 times the Jarlskog invariant times the Fermi constant. This is much larger than predicted from Jarlskog determinant considerations based on perturbation theory and is fully in agreement with the expectations first put forward by Smit in [11].

Remarkably the non vanishing CP violating contributions come from the normal parity sector. This is somewhat unexpected as it is usually taken for granted that the presence of the Levi-Civita pseudo-tensor is needed to have CP breaking. Also noteworthy is our finding that, to the order studied, all abnormal parity terms vanish. This is in conflict with

the result presented in [12] where a non vanishing abnormal parity contribution is derived. Our result implies that the first CP odd and P odd contribution, relevant to electric dipole moments, requires at least the eighth order in the derivative expansion.

The full result presents interesting regularities (including the just mentioned vanishing of terms which are simultaneously CP odd and parity odd) which may follow from the structure of the SM or may be accidental symmetries surviving only at lowest orders in the derivative expansion.

The calculation presented here applies to zero temperature. At the order studied in this work, the coupling is controlled by the integral $\hat{I}^3_{1,1,2,2}$. The same integral can be considered at finite temperature replacing the energy integral in the loop by a fermionic Matsubara sum. At high enough temperatures such integral becomes proportional to the Jarlskog determinant times the accompanying power $1/T^{14}$, just by dimensional counting. For temperatures comparable to the top quark mass, relevant for baryogenesis, this mass can no longer be treated as a perturbation. Nevertheless, the integral becomes equal to the Jarlskog determinant times an order of unity function of m_t/T and the required power $1/T^{14}$, so the numerical estimate is similarly small. At higher order in the derivative expansion other momentum integrals will be generated but the analysis is expected to be similar. On the other hand, at finite temperature new operators can be produced even at the same order studied here. Due the to breaking of Lorentz invariance down to rotational invariance [32] new combinations may arise, e.g., with abnormal parity.

The effective action obtained here can not be applied directly to CP violating hadronic processes since the quarks have been integrated out. In this view it would be of interest to repeat the calculation but inserting external hadronic currents in the quark sector. This would allow to account for CP violating contributions to meson decay amplitudes. CP violation requires to switch between the u and d spaces and in the present calculation this can only be achieved by the action of $W^\pm$ which are the only charged particles left after integration of fermions. The situation may change in the presence of hadronic insertions carrying electric charge, and presumably lower order operators would be allowed replacing some charged gauge bosons by hadronic currents.

Related to the above, we have seen that the coupling $\kappa_{\rm CP}$ controlling the strength of the CP violating operators is a complicated function of the quark masses and is highly non continuous in the relevant limit of taking the light quarks to be massless. This is due to the presence of infrared divergences in the chiral limit. This suggests that the standard chiral perturbation theory corrections induced by the pseudo-scalar Goldstone bosons could introduce sizable modifications to the result. Formally, we have included QCD in our result, although no gluonic correction have been needed to the order considered. Of course, the calculation can be organized under different schemes, for instance counting independently derivatives coming from electroweak fields and QCD fields, being the latter either fundamental or hadronic. A point to note in this regard is that in the presence of hadrons abnormal parity is no longer equivalent to parity violating, as the pseudo-scalars, for instance, have abnormal parity.

Likewise it would be of interest to consider the effective action in the sense of the Legendre transform (after integration of fermions), i.e., adding one particle irreducible

graphs. This could produce lower dimensional CP breaking operators, such as the operator $\text{tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu})$ mentioned in the Introduction. This can be so by integration of some of the external lines in the loop (hence making these lines internal) either in a single fermion loop having CP violation or by coupling more than one loop, one of them carrying CP violation.

As a final comment we note that leptons have not been included as no CP breaking takes place in that sector when the neutrino masses vanish. However, our calculation applies to Dirac neutrinos with non zero masses, and in this case the leptons produce a CP violating contribution completely similar to that obtained for quarks, replacing the Jarlskog invariant of the CKM matrix with that of the Maki-Nakagawa-Sakata matrix [36], and using lepton masses to produce $\kappa_{\text{CP}}^{\text{leptons}}$. This function will be identical to that discussed in section 11 for quarks. Similarly as for the case of quarks, the coefficient $\kappa_{\text{CP}}^{\text{leptons}}$ is not at all continuous as some of the leptons become massless. This implies that the limit will depend crucially on how the small neutrino masses compare with each other.

Acknowledgments

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